Equivalent EMF and IR of a Battery Pack

How is the equivalent electromotive force and internal resistance of a battery pack influenced by the arrangement (series, parallel, or hybrid) and the own electromotive forces and internal resistances of its component cells?

Subject: Physics
Word Count: 3966

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1 Introductions

1.1 Introduction to this Essay

In this extended essay, I intend to investigate on the relationship between the equivalent electromotive force (henceforward referred to as EMF) and the internal resistance (henceforward referred to as IR) of a battery pack, and the EMF and IR of its component batteries.

I will first derive expressions for the equivalent EMF and IR of a battery pack, with respect to the EMF and IR of its component batteries and their connection types, using the basic electrical laws. I will then verify my results experimentally. Finally, I will give an evaluation of my research between my theory predictions and experimental results.

1.2 My Rationale on the Topic of this Essay

During my curious explorations of the electrical circuits, I know that the equivalent EMF of a battery pack, connected in series, is the sum of the EMF of its component batteries; and I know that the equivalent EMF of a battery pack of same type, connected in parallel, is just the EMF of any of its component batteries; but I never have an idea of the EMF and IR of a battery pack of different types, or even connected in complex ways of connection. However, with the electrical theories I learned in my IBDP physics class, I am able and willing to explore the EMF and IR of packs of batteries in a complex connection type, and to explore the knowledge behind the traditional lectures and beyond the course syllabus.

1.3 Table of Frequently Used Variables

I will use table 1 as a standard specification for the variables.

Table 1: Frequently used variables

Variable	Description	Unit
$\mathcal{E}$	Electromotive force of a battery	Volt
$\mathcal{E}_{eq}$	Equivalent EMF of a battery pack	Volt
I	Current in a circuit	Ampere
R	Resistance of a resistor in a circuit	Ohm
r	Internal resistance of a battery	Ohm
r_{eq}	Equivalent IR of a battery pack	Ohm
n	Total number of currents, loops, or batteries	n.a.
i	Index of currents, loops, or batteries	n.a.
k	Case index of mathematical inductions	n.a.

2 Theories

2.1 Definition of Equivalent EMF and IR

Consider figure 1. If there is a single battery whose EMF and IR equals to the EMF and IR of a battery pack, then the equivalent EMF and IR of the battery pack is defined to be the EMF and IR of that single battery.

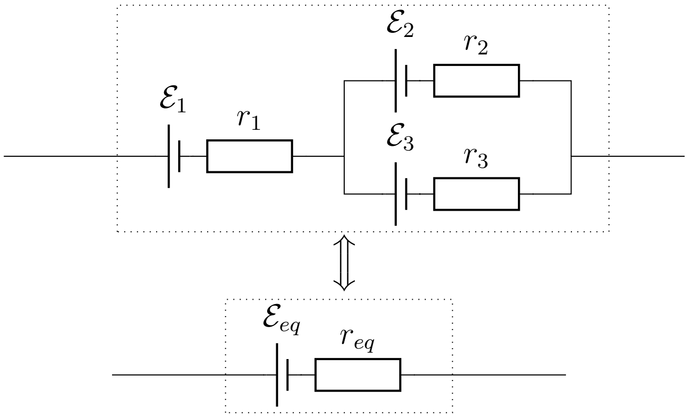


Figure 1: Equivalent EMF and IR of a battery pack

2.2 Elementary Relationships in Electrical Circuits

The following three elementary relationships are used as lemmas in the derivation for expressions of the equivalent EMF and IR of a battery pack.

2.2.1 The Kirchhoff's Circuit Laws

The Junction Rule Consider any junction in a circuit. The sum of the electrical currents coming to and going away from junction equals to zero (Oldham, 2008):

$$\sum_{i=1}^{n} I_i = 0. \tag{1}$$

The Loop Rule Consider any loop in a circuit. The sum of the potential differences across the electrical appliances around the loop equals to zero (Oldham, 2008):

$$\sum_{i=1}^{n} V_i = 0. \tag{2}$$

2.2.2 EMF and IR of a Battery in a Resistor Circuit

Putting a battery into a circuit with an arbitrary resistor, shown in figure 2, then there is an expression of the EMF and IR of that battery.

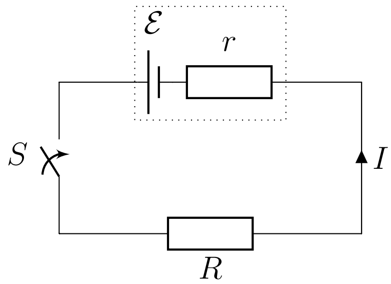


Figure 2: A battery in a resistor circuit

According to the Loop Rule of the Kirchhoff's Circuit Laws (equation (2)),

$$\mathcal{E} = IR + Ir. \tag{3}$$

It could be expressed as

$$r = \frac{\mathcal{E}}{I} - R. \tag{4}$$

This is a formula to calculate the IR of a battery.

2.2.3 EMF and Terminal Voltage of a Battery in a Resistor Circuit

When the arbitrary resistor R in figure 2 approaches to infinity, equation (3) becomes

$$\lim_{R \to \infty \Omega} \mathcal{E} = \lim_{R \to \infty \Omega} (IR + Ir). \tag{5}$$

In the equation, the terminal voltage of the battery is IR. As the arbitrary resistor R in the circuit approaches to infinity, the current I in the circuit approaches to zero. As r is finite, the second term of the right hand side goes away. Simplifying the equation, I get

$$\mathcal{E} = \lim_{R \to \infty \Omega} IR. \tag{6}$$

In other words, the EMF of a battery equals to its terminal voltage in an **open** circuit. This is a formula to calculate the EMF of a battery.

2.3 Equivalent EMF and IR in Series Connections

2.3.1 Equivalent EMF with Two Component Batteries

Similar as shown in figure 2, I put a battery pack into a resistor circuit and draw a more detailed circuit in figure 3.

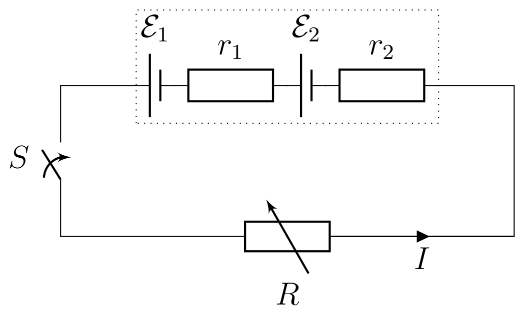


Figure 3: A battery pack consists of two component batteries, connected in series, in a resistor circuit

Applying the Loop Rule of Kirchhoff's Circuit Laws (equation (2)) for the main loop of the circuit in figure 3, I get

$$\mathcal{E}_1 + \mathcal{E}_2 = Ir_1 + Ir_2 + IR. \tag{7}$$

Expressing the current I using the rest of the variables, I transformed the previous equation to

$$I = \frac{\mathcal{E}_1 + \mathcal{E}_2}{r_1 + r_2 + R}. \tag{8}$$

Since the equivalent EMF of this battery pack is its terminal voltage in an open circuit, I expressed it by plugging the expression of I into equation (6):

$$\begin{aligned} \mathcal{E}_{eq,2} &= \lim_{R \to \infty \Omega} \frac{\mathcal{E}_1 + \mathcal{E}_2}{r_1 + r_2 + R} R && (9) \\ &= \mathcal{E}_1 + \mathcal{E}_2. && (10) \end{aligned}$$

2.3.2 Equivalent IR with Two Component Batteries

By starring at the circuit in figure 3, you may guess that the equivalent IR of that battery pack is the sum of the IR of its component batteries, and that is true. By plugging the expression for the equivalent EMF in equation (10) into equation (4), I get

$$r_{eq,2} = \frac{\mathcal{E}_1 + \mathcal{E}_2}{I} - R. \tag{11}$$

Then I plug in the expression for current I from equation (7) to the previous equation to get an expression for the equivalent IR:

$$r_{eq,2} = \frac{\mathcal{E}_1 + \mathcal{E}_2}{\frac{\mathcal{E}_1+\mathcal{E}_2}{r_1+r_2+R}} - R \tag{12}$$

$$= r_1 + r_2. \tag{13}$$

2.3.3 Equivalent EMF with Multiple Component Batteries

Consider figure 4. For a pack of battery, there is no difference between the situation where some of its component batteries are connected in series and then connected to another battery, and the situation where all of its component batteries are connected in series equally.

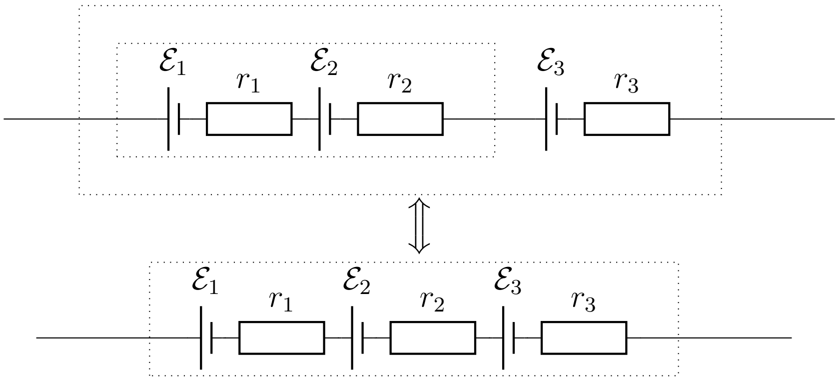


Figure 4: Encapsulation of batteries connected in series has **no** effect

As a result, I hypothesize that the equivalent EMF of a battery pack is the sum of the EMF of its individual batteries:

$$\mathcal{E}_{eq,n} = \sum_{i=1}^{n} \mathcal{E}_i. \tag{14}$$

I use the expression of the equivalent EMF in equation (10) as a base case consist of two component batteries. Then, if my hypothesis is true for the kth case,

$$\mathcal{E}_{eq,k} = \sum_{i=1}^{k} \mathcal{E}_i, \tag{15}$$

then it must be true for the $(k+1)$th case as well, by mathematical induction:

$$\begin{aligned} \mathcal{E}_{eq,k+1} &= \mathcal{E}_{eq,k} + \mathcal{E}_{k+1} && (16) \\ &= \sum_{i=1}^{k} \mathcal{E}_i + \mathcal{E}_{k+1} && (17) \\ &= \sum_{i=1}^{k+1} \mathcal{E}_i. && (18) \end{aligned}$$

So equation (14) is a correct general expression for the equivalent EMF of a battery pack with its component batteries connected in series.

2.3.4 Equivalent IR with Multiple Component Batteries

Similarly, I hypothesize that the equivalent IR of a battery pack is the sum of the IR of its individual batteries:

$$r_{eq,n} = \sum_{i=1}^{n} r_i. \tag{19}$$

And analogously, using equation (13) as a base case, then if my hypothesis is true for the kth case,

$$r_{eq,k} = \sum_{i=1}^{k} r_i, \tag{20}$$

then it must be true for the $(k+1)$th case as well, by mathematical induction:

$$r_{eq,k+1} = r_{eq,k} + r_{k+1} \tag{21}$$
$$= \sum_{i=1}^{k} r_i + r_{k+1} \tag{22}$$
$$= \sum_{i=1}^{k+1} r_i. \tag{23}$$

So equation (19) is a correct general expression for the equivalent IR of a battery pack with its component batteries connected in series.

2.4 Equivalent EMF and IR in Parallel Connections

2.4.1 Equivalent EMF with Two Component Batteries

Similar as shown in figure 2, I put a battery pack into a resistor circuit and draw a more detailed circuit in figure 5.

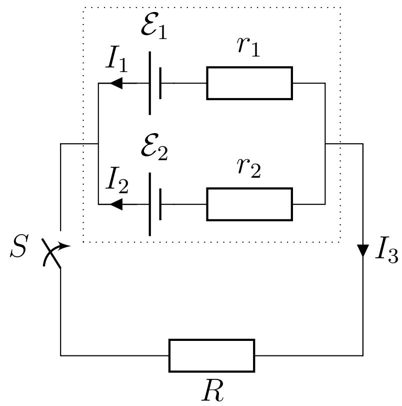


Figure 5: A battery pack consists of two component batteries, connected in parallel, in a resistor circuit

This circuit looks more complex. Applying the Loop Rule of the Kirchhoff's Circuit Laws (equation (2)) to the two loops of this circuit, I get equation (24) and equation (25):

$$-I_1r_1 + \mathcal{E}_1 - \mathcal{E}_2 + I_2r_2 = 0, \tag{24}$$

$$-I_2r_2 + \mathcal{E}_2 - I_3R = 0. \tag{25}$$

Similarly, applying the Junction Rule of the Kirchhoff's Circuit Laws (equation (1)) to the junction at the right of internal resistor r_2, I get

$$I_1 + I_2 = I_3. \tag{26}$$

Solving the above simultaneous equations (24) (25) (26), I have an expression for the current in the main circuit I_3:

$$I_3 = \frac{\mathcal{E}_1r_2 + \mathcal{E}_2r_1}{r_1r_2 + Rr_1 + Rr_2}. \tag{27}$$

Recalling that the equivalent EMF of this battery pack is its terminal voltage in an open circuit, I expressed it by replacing the current I in equation (6) by current in the main circuit I_3, according to equation (27):

$$\mathcal{E}_{eq,2} = \lim_{R\to\infty\Omega} \frac{\mathcal{E}_1r_2 + \mathcal{E}_2r_1}{r_1r_2 + Rr_1 + Rr_2}R \tag{28}$$

$$= \frac{r_2}{r_1 + r_2}\mathcal{E}_1 + \frac{r_1}{r_1 + r_2}\mathcal{E}_2. \tag{29}$$

2.4.2 Equivalent IR with Two Component Batteries

Similarly, by plugging the expression of equivalent EMF in equation (28) and the expression of the current in the main circuit I_3 into equation (4), I have an expression for equivalent IR:

$$r_{eq,2} = \frac{\dfrac{r_2}{r_1+r_2}\mathcal{E}_1 + \dfrac{r_1}{r_1+r_2}\mathcal{E}_2}{\dfrac{\mathcal{E}_1 r_2 + \mathcal{E}_2 r_1}{r_1 r_2 + R r_1 + R r_2}} - R \tag{30}$$

$$= \frac{r_1 r_2}{r_1 + r_2}. \tag{31}$$

Property of these Expressions Notice that a property is given from the symmetry of my expressions, that the arrangement of the component batteries in a battery pack does **not** influence its equivalent EMF and IR. This property affirms the validity of my expression, because there is, in fact, **no** arrangement for the component batteries connected in parallel: all the anodes of the component batteries are connected together into one, and all the cathodes of the component batteries are connected together into one as well. It is unlike the situation where all the batteries in a battery pack is connected in series, where there **is** certain number of arrangements of the component batteries. (Although the equivalent EMF and IR of the battery pack is, still, **not** influenced by the arrangement of its component batteries.)

2.4.3 Equivalent IR with Multiple Component Batteries

Consider equation (31). By taking the inverse on the both sides of the equation, I find a simpler expression for the equivalent IR:

$$\frac{1}{r_{eq,2}} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{32}$$

Notice that this expression is linear with respect to the inverse of IR of the component batteries. In the situation where the component batteries of a battery pack are connected in parallel, the linearity of the expression of the equivalent IR and EMF is hidden in the seemingly complex formula, but by digging out such linearity, I could conduct mathematical inductions easier.

And what's more, similar as figure 4, consider figure 6. For a pack of battery, there is no difference between the situation where some of its component batteries are connected in parallel and then connected to another battery, and the situation where all of its component batteries are connected in parallel equally as well.

So I hypothesize that the inverse of the equivalent IR of a battery pack is equal to the sum of the inverse of the equivalent IR of its component batteries:

$$\frac{1}{r_{eq,n}} = \sum_{i=1}^{n} \frac{1}{r_i}, \tag{33}$$

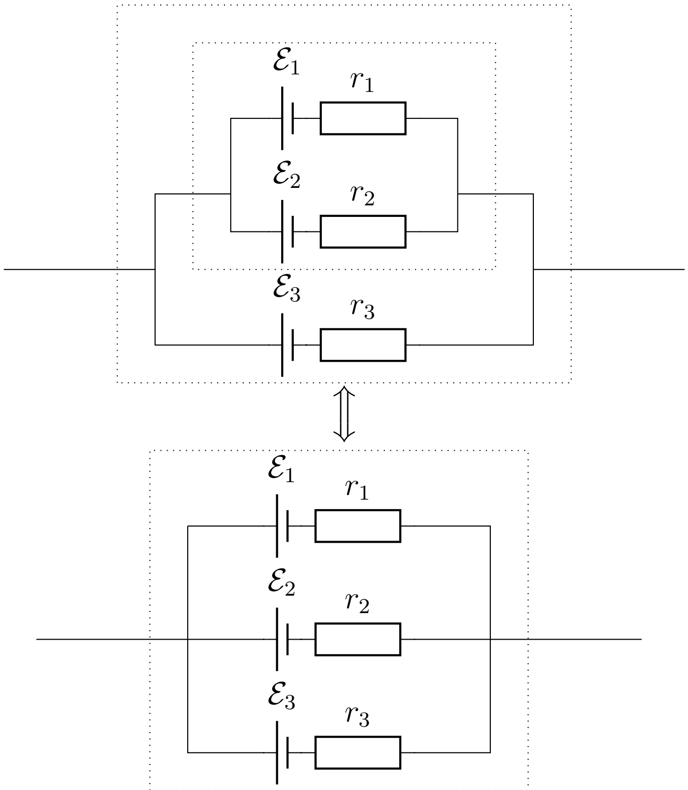


Figure 6: Encapsulation of batteries connected in parallel has **no** effect

or even more explicitly (but a bit more complex) as

$$r_{eq,n} = \frac{1}{\displaystyle\sum_{i=1}^{n} \frac{1}{r_i}}. \tag{34}$$

I could use equation (32) as a base case. Then if my hypothesis is true for the kth case,

$$\frac{1}{r_{eq,k}} = \sum_{i=1}^{k} \frac{1}{r_i}, \tag{35}$$

then it must be true for the $(k+1)$th case as well, by mathematical induction:

$$\frac{1}{r_{eq,k+1}} = \frac{1}{r_{eq,k}} + \frac{1}{r_{k+1}} \tag{36}$$

$$= \sum_{i=1}^{k} \frac{1}{r_i} + \frac{1}{r_{k+1}} \tag{37}$$

$$= \sum_{i=1}^{k+1} \frac{1}{r_i}. \tag{38}$$

So equation (34), following from equation (33), is a correct general expression for the equivalent IR of a battery pack with its component batteries connected in parallel.

2.4.4 Equivalent EMF with Multiple Component Batteries

Similarly, from equation (29), I find a simpler expression for the equivalent EMF of a battery pack consists of two component batteries, formatted in equation (30):

$$\mathcal{E}_{eq,2} = \frac{r_2 r_1}{r_1 + r_2} \frac{\mathcal{E}_1}{r_1} + \frac{r_1 r_2}{r_1 + r_2} \frac{\mathcal{E}_2}{r_2} \tag{39}$$

$$= r_{eq,2} \left(\frac{\mathcal{E}_1}{r_1} + \frac{\mathcal{E}_2}{r_2} \right). \tag{40}$$

Notice that this expression is linear, with respect to the **quotient** of the EMF and IR of the component batteries, as well.

As a result, I hypothesize that the equivalent EMF of a battery pack is equal to the equivalent IR of the battery pack times the sum of quotient of the

EMF and IR of the component batteries:

$$\mathcal{E}_{eq,n} = r_{eq,n} \cdot \sum_{i=1}^{n} \frac{\mathcal{E}_i}{r_i}. \tag{41}$$

I could use equation (40) as a base case. Then, if my hypothesis is true for kth case,

$$\mathcal{E}_{eq,k} = r_{eq,k} \cdot \sum_{i=1}^{k} \frac{\mathcal{E}_i}{r_i}, \tag{42}$$

then it must be true for the $(k+1)$th case as well by mathematical induction:

$$\begin{aligned}
\mathcal{E}_{eq,k+1} &= \frac{1}{\dfrac{1}{r_{eq,k}} + \dfrac{1}{r_{k+1}}} \left(\frac{\mathcal{E}_{eq,k}}{r_{eq,k}} + \frac{\mathcal{E}_{k+1}}{r_{k+1}} \right) && (43) \\
&= r_{eq,k+1} \left(\sum_{i=1}^{k} \frac{\mathcal{E}_i}{r_i} + \frac{\mathcal{E}_{k+1}}{r_{k+1}} \right) && (44) \\
&= r_{eq,k+1} \cdot \sum_{i=1}^{k+1} \frac{\mathcal{E}_i}{r_i}. && (45)
\end{aligned}$$

So equation (41) is a correct general expression for the equivalent EMF of a battery pack with its component batteries connected in parallel.

3 Experiments

3.1 General Ideas of my Experiment Design

I measure the EMF and IR of a battery or a battery pack by connecting it according to the circuit in figure 7, and analyze the readings in the voltmeter and ammeter. The voltmeter in the circuit measures the terminal voltage of the battery, while the ammeter measure the current passing through the battery. These readings on the voltmeter and ammeter will change, so I could measure for several pairs of data.

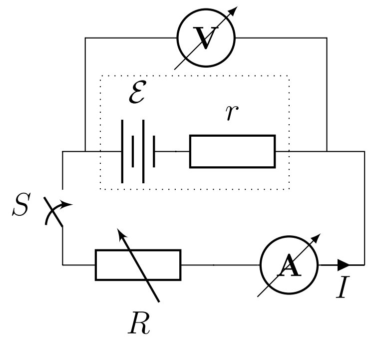


Figure 7: Measuring circuit on the EMF and IR of a battery

Recall that from equation (3), I have

$$\mathcal{E} = I(R + r). \tag{46}$$

In addition, applying the Loop Rule of the Kirchhoff's Circuit Laws (equation (2)) to the upper loop in the circuit, I get

$$V = IR. \tag{47}$$

Combining equation (46) and equation (47), I get an equation of which V is expressed linearly by I:

$$V = -Ir + \mathcal{E}. \tag{48}$$

So V and I are linearly dependent. Since EMF and IR are the only two constants in this expression, then if I scatter several pairs of V and I onto a graph and find the line of best fit using univariate linear regression, then the y-intercept of the line of best fit is the EMF of the battery, and the negative slope of the line of best fit is the IR of the battery. (Tsokos, 2014)

3.2 Experiment Environments

I did my experiments in the laboratory in my school, using provided gadgets in the laboratary. A photo of my experiment environments is provided in figure 8.

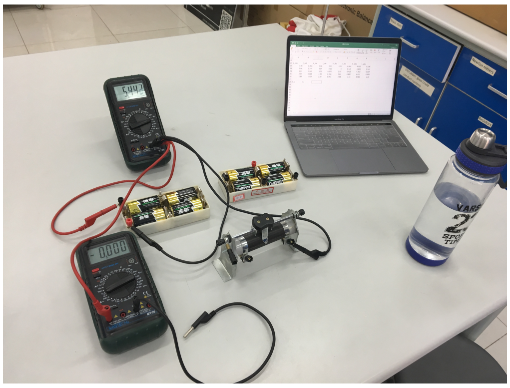


Figure 8: Experiment environments

The connected circuit for the experiment is shown in figure 9. I didn't intentionally use a switch in my circuit, but instead unplug the wire as a switch from a point of circuit, so that I can switch the circuit faster and more conveniently.

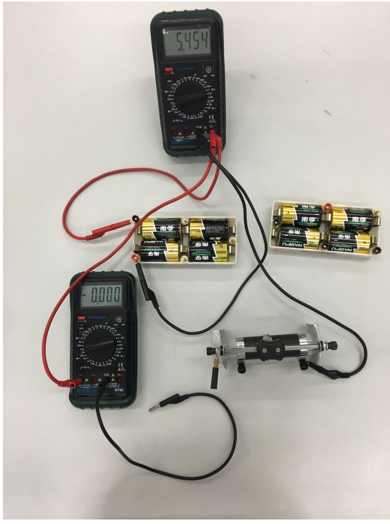


Figure 9: Real circuit to measure the EMF and IR of a battery

3.3 Experiment Group Settings

3.3.1 Connection of Batteries with Same EMF and IR

I conducted multiple sets of experiments. Each set of the experiment measures five pairs of V and I, such that I could calculate the equivalent EMF and IR of the measured battery, according to equation (48), by plot those pairs on a graph and find the line of best fit using linear regression.

I used battery A, B and C, which are standard $1.5V$ D-size batteries. I first plugged each of the batteries individually into the measuring circuit to get three sets of data. Then I combined each pair of battery A, B and C, in both series and parallel connections, as battery packs, and plugged them into the measuring equations to get six more sets of data. Finally, I combined all three of the battery A, B, and C, in both series and parallel connections, to form battery packs, and plugged the them into the measuring circuit to get another two sets of data.

In these first sets of experiments, I connected batteries of same EMF and IR for safety concerns. By using batteries of small and same voltages, I could test my proposed theory experimentally safely.

3.3.2 Connection of Batteries with Different EMF and IR

I used battery D, E, F, G, H, I, J, K, and L in my later experiments. These batteries are composed of several new standard 1.5 V D-sized batteries. Similarly, I plugged in each of these combined batteries, from battery D to L, to the measuring circuit, to get several sets of data. Then I make battery packs using these combined batteries, so the component batteries of my battery pack could have different EMF and IR. I then connected these combined batteries in both series and parallel connections, and plugged the battery pack which consists of them into my measuring circuit, to get the last several sets of data.

These further sets of experiments are used to verify my theory under the condition which the component batteries of a battery pack have different EMF and IR.

I planned to have several sets of experiment on the battery packs consisting of **multiple** batteries, connected in both series and parallel, but didn't finish them because of my limited time in laboratory.

3.4 Step-wised Experiment Instructions

1. Connect the circuit according to figure 7 while leaving the battery part to be empty. The variable resistor is set to maximum resistance for safety. Turn on the multimeters and set them to proper modes.

2. Plug in a battery pack into the measuring circuit. There should now be readings on the voltmeter and ammeter.

3. Lower the resistance of the variable resistor a little bit.

4. Record V and I from the voltmeter and ammeter respectively.

5. Repeat step 5 and 6 four more times to record five pairs of readings in total.

4 Data Analysis

In this section, I will calculate the **theoretical** equivalent EMF and IR according to my proposed theory, using the measured EMF and IR of the component batteries of a battery pack. I will also calculate the **real** equivalent EMF and IR measured directly from the battery pack. The **error** in them is then given in percentage.

In the figures of this section, each of the two graphs on the left is a plot of pairs of V and I from a set of data from the **component** batteries of a battery pack. On the other hand, the graph on the right is a plot of pairs of V and I from a set of data from the battery pack itself.

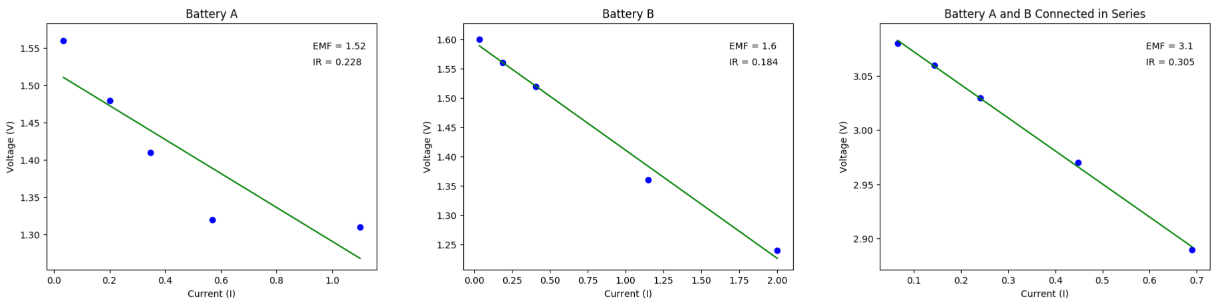
4.1 Series Connections

4.1.1 Connection with Two Component Batteries of Same EMF and IR

Here, the predicted (theoretical) EMF of battery packs are very accurate, for all the errors in EMF are below five percent.

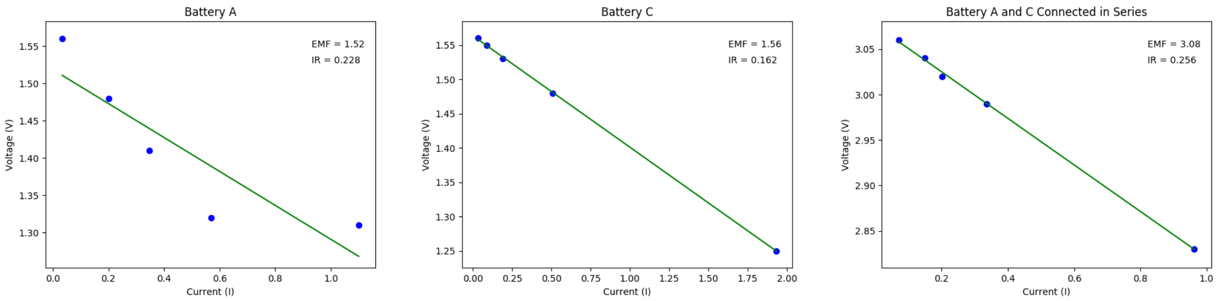
However, the predicted IR of the battery packs are less accurate, with an average error in IR above thirty percent, and a maximum error in IR of 52.3 percent in figure 11.

Looking at the real IR in the table, I realized that the multimeters are not really accurate enough to detect resistance to 0.001 Ω. As a result, only the order of magnitude of the real equivalent IR is predicted accurately.



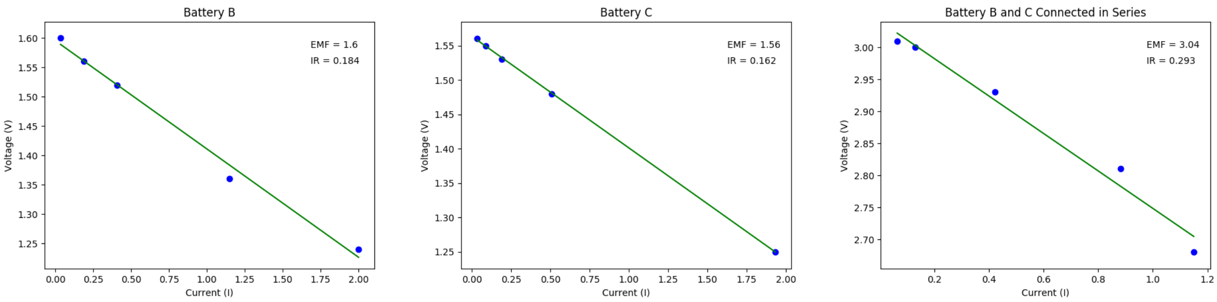
Theoretical EMF	3.11	V	Real EMF	3.10	V	Error in EMF	0.347	%
Theoretical IR	0.412	Ω	Real IR	0.305	Ω	Error in IR	35.2	%

Figure 10: A battery pack consists of battery A and B connected in Series



Theoretical EMF	3.08	V	Real EMF	3.08	V	Error in EMF	0.188	%
Theoretical IR	0.39	Ω	Real IR	0.256	Ω	Error in IR	52.3	%

Figure 11: A battery pack consists of battery A and C connected in Series



Theoretical EMF	3.16	V	Real EMF	3.04	V	Error in EMF	3.87	%
Theoretical IR	0.347	Ω	Real IR	0.293	Ω	Error in IR	18.5	%

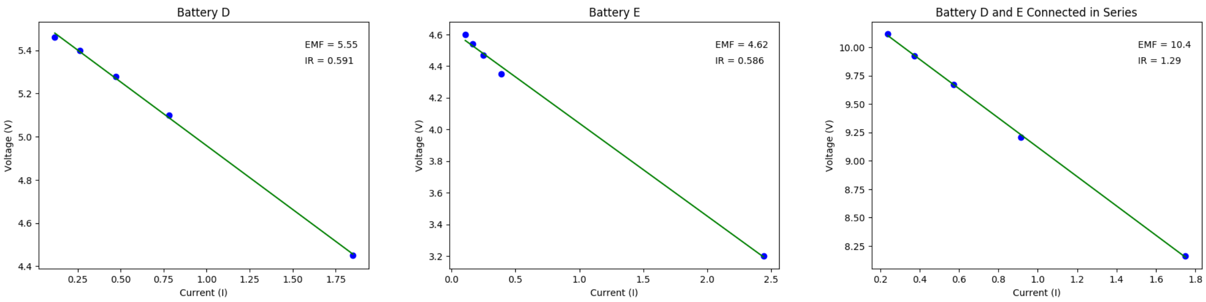
Figure 12: A battery pack consists of battery B and C connected in Series

4.1.2 Connection with Two Component Batteries of Different EMF and IR

I tried several sets of data, where the EMF and IR of the component batteries in a battery pack are different.

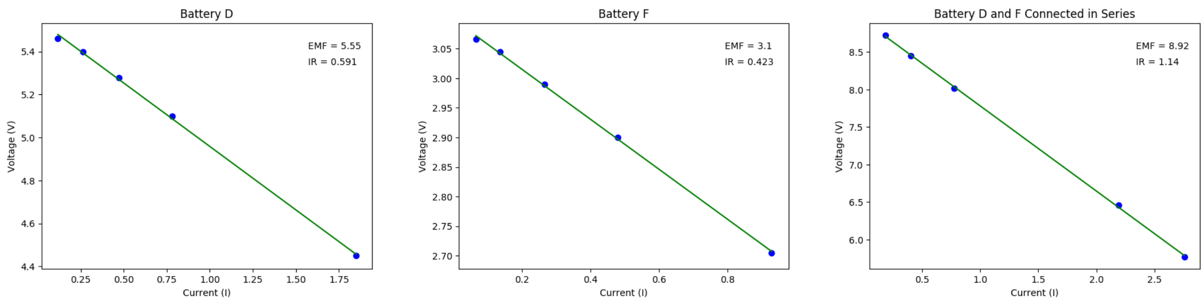
The theoretical EMF remains to be accurately predicted, with all the errors in EMF below 5 percent. On the other hand, the theoretical IR is predicted more accurately, for the majority of the errors in IR below or around 10 percent. This is because that an increase in the IR of the battery pack and its component batteries leads to less percentage errors of those measured IR.

I noticed an exceptional high error in IR in figure 17, with its error in IR as high as 26.4 percent. Noticing that the pairs of V and I in the graph of battery J doesn't fall nicely onto its line of best fit. This might be an explanation for its exceptional high source of error.



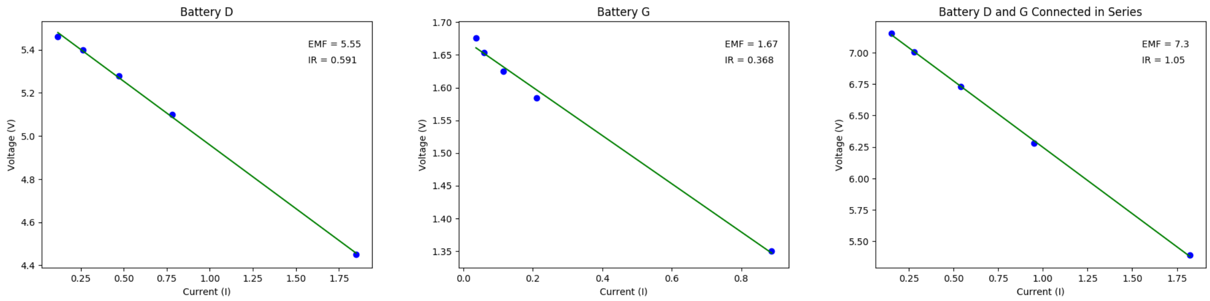
Theoretical EMF	10.2	V	Real EMF	10.4	V	Error in EMF	2.28	%
Theoretical IR	1.18	Ω	Real IR	1.29	Ω	Error in IR	8.88	%

Figure 13: A battery pack consists of battery D and E connected in Series



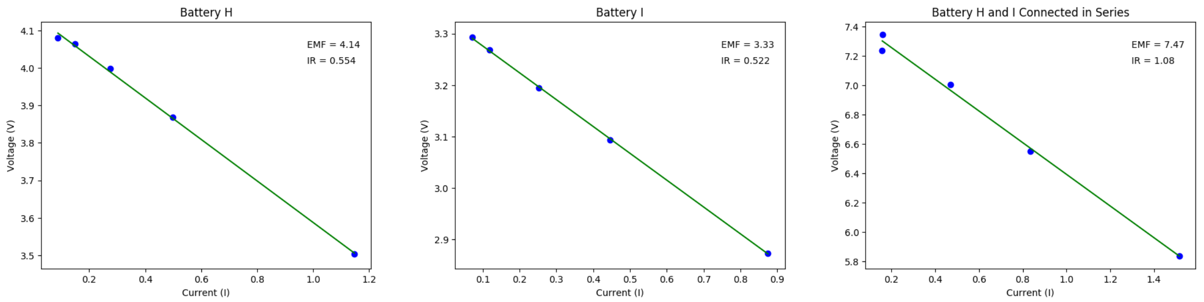
Theoretical EMF	8.65	V	Real EMF	8.92	V	Error in EMF	3.03	%
Theoretical IR	1.01	Ω	Real IR	1.14	Ω	Error in IR	10.8	%

Figure 14: A battery pack consists of battery D and F connected in Series



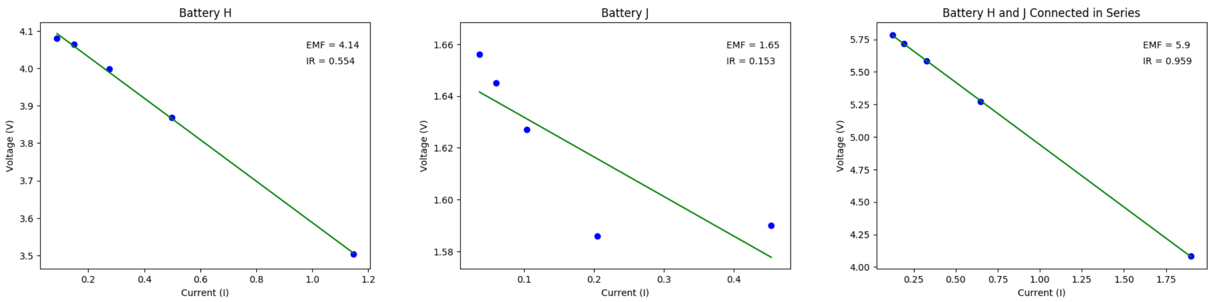
Theoretical EMF	7.22	V	Real EMF	7.30	V	Error in EMF	1.07	%
Theoretical IR	0.959	Ω	Real IR	1.05	Ω	Error in IR	9.00	%

Figure 15: A battery pack consists of battery D and G connected in Series



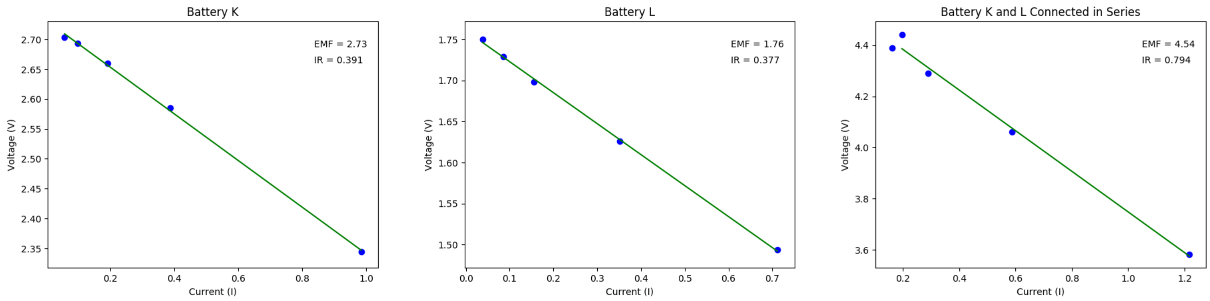
Theoretical EMF	7.47	V	Real EMF	7.47	V	Error in EMF	0.0282	%
Theoretical IR	1.08	Ω	Real IR	1.08	Ω	Error in IR	0.228	%

Figure 16: A battery pack consists of battery H and I connected in Series



Theoretical EMF	5.79	V	Real EMF	5.90	V	Error in EMF	1.87	%
Theoretical IR	0.706	Ω	Real IR	0.959	Ω	Error in IR	26.4	%

Figure 17: A battery pack consists of battery H and J connected in Series

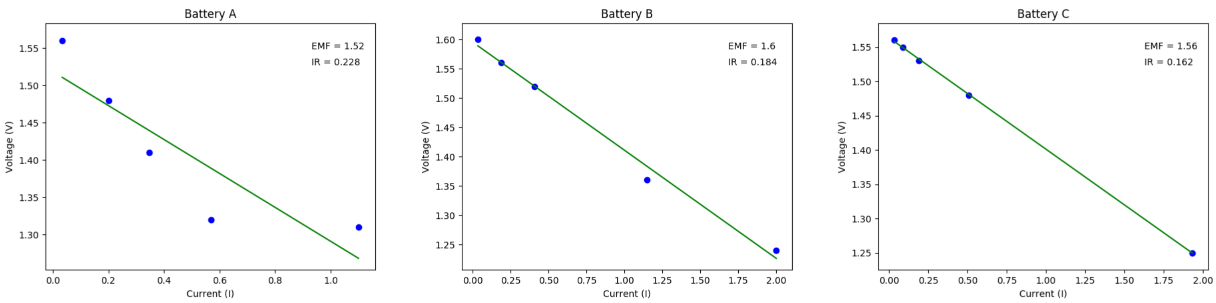


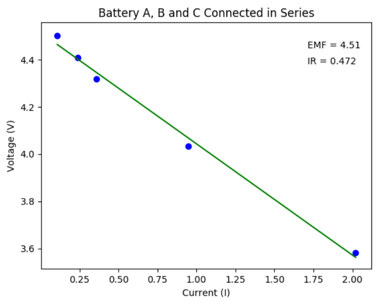
Theoretical EMF	4.49	V	Real EMF	4.54	V	Error in EMF	1.09	%
Theoretical IR	0.768	Ω	Real IR	0.794	Ω	Error in IR	3.34	%

Figure 18: A battery pack consists of battery K and L connected in Series

4.1.3 Connection with Multiple Component Batteries

In figure 19, the theoretical EMF still remains to be accurately predicted, while the theoretical IR still has a high error of 21.8 percent, due to a very small value of the IR of the measured battery pack and its component batteries.





Theoretical EMF	4.68	V	Theoretical IR	0.575	V
Real EMF	4.51	Ω	Real IR	0.472	Ω
Error in EMF	3.59	%	Error in IR	21.8	%

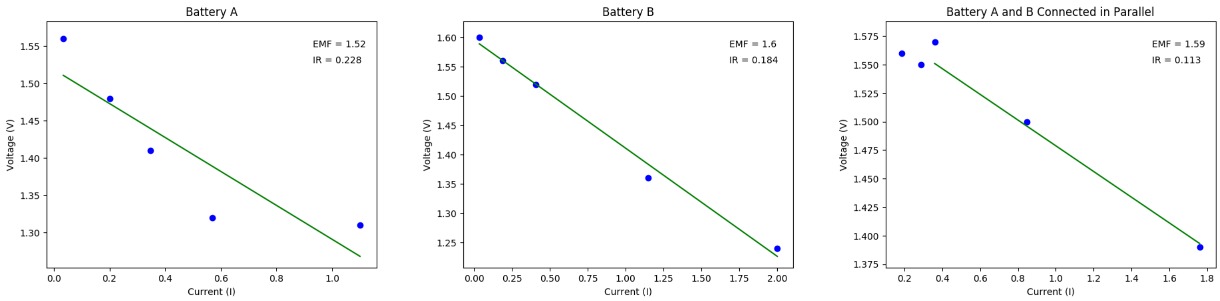
Figure 19: A battery pack consists of battery A, B and C connected in Series

4.2 Parallel Connections

4.2.1 Connection with Two Component Batteries of Same EMF and IR

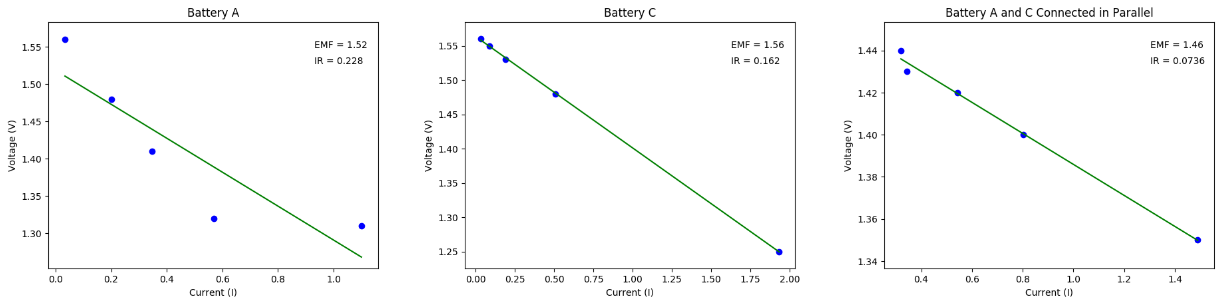
Similarly, by connecting each pair of the three batteries, A, B, and C, in parallel, I obtain the following graph of the battery.

While the predicted EMF remains accurate with a general error below five percent, the predicted IR is still not accurate, with a general error of 30 percent.



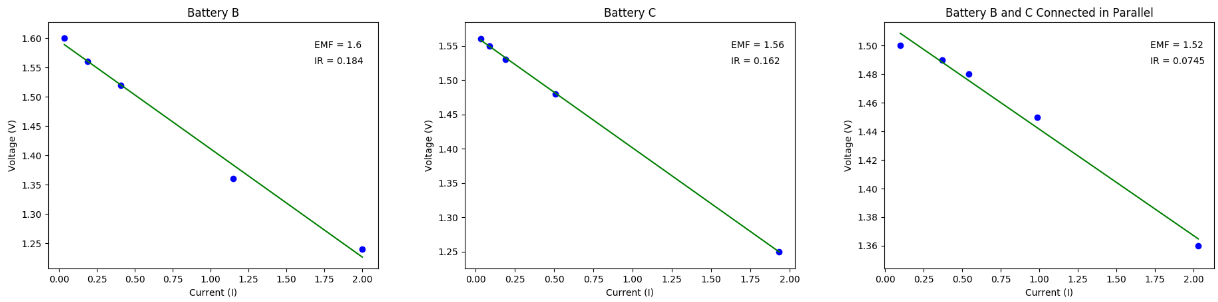
Theoretical EMF	1.56	V	Real EMF	1.59	V	Error in EMF	1.93	%
Theoretical IR	0.102	Ω	Real IR	0.113	Ω	Error in IR	9.63	%

Figure 20: A battery pack consists of battery A and B connected in Parallel



Theoretical EMF	1.54	V	Real EMF	1.46	V	Error in EMF	5.84	%
Theoretical IR	0.0948	Ω	Real IR	0.0736	Ω	Error in IR	28.8	%

Figure 21: A battery pack consists of battery A and C connected in Parallel



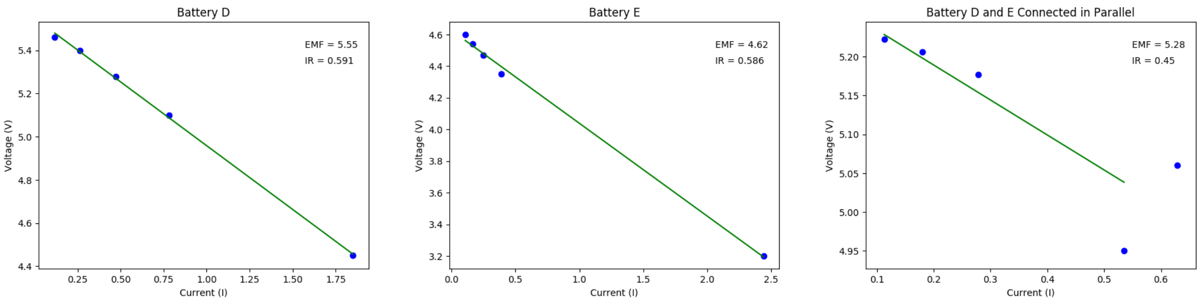
Theoretical EMF	1.58	V	Real EMF	1.52	V	Error in EMF	4.11	%
Theoretical IR	0.0863	Ω	Real IR	0.0745	Ω	Error in IR	15.8	%

Figure 22: A battery pack consists of battery B and C connected in Parallel

4.2.2 Connection with Two Component Batteries of Different EMF and IR

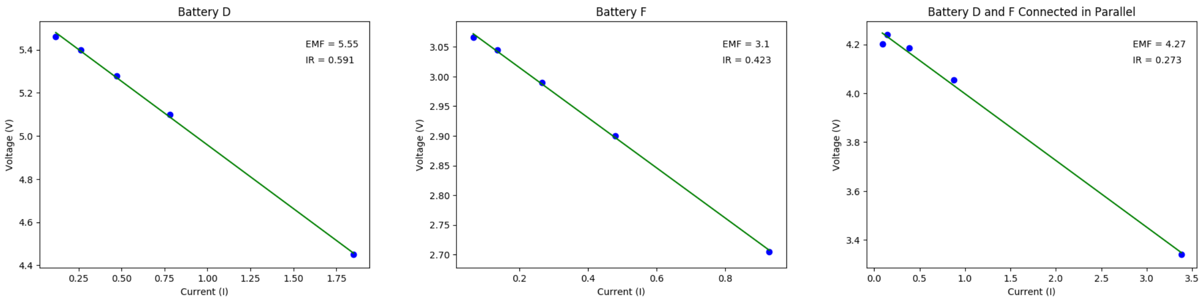
By connecting the batteries of different EMF and IR in parallel, I obtain the following results.

Here the error increases. While the errors in EMF is around 10 percent, the errors in IR has increased dramatically with a maximum value of 50 percent. A possible explanation is that since the IR of a battery pack is even less than the IR of its component batteries in parallel, the measured IR of a battery has even larger percentage error, leading to a larger error of its predictions.



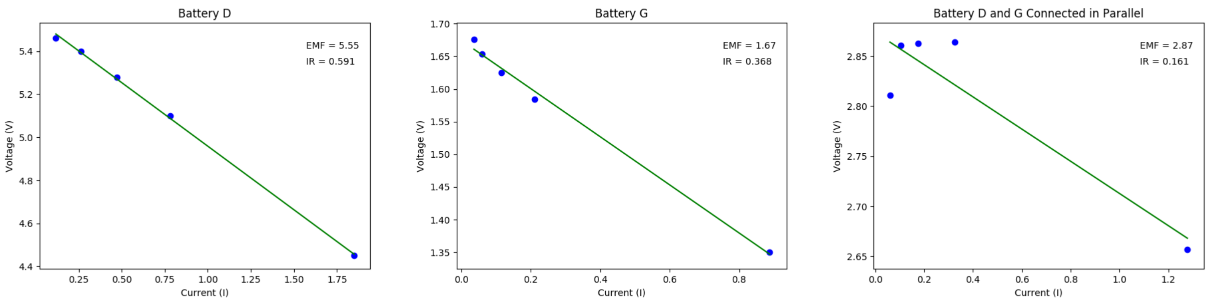
Theoretical EMF	5.09	V	Real EMF	5.28	V	Error in EMF	3.67	%
Theoretical IR	0.294	Ω	Real IR	0.450	Ω	Error in IR	34.6	%

Figure 23: A battery pack consists of battery D and E connected in Parallel



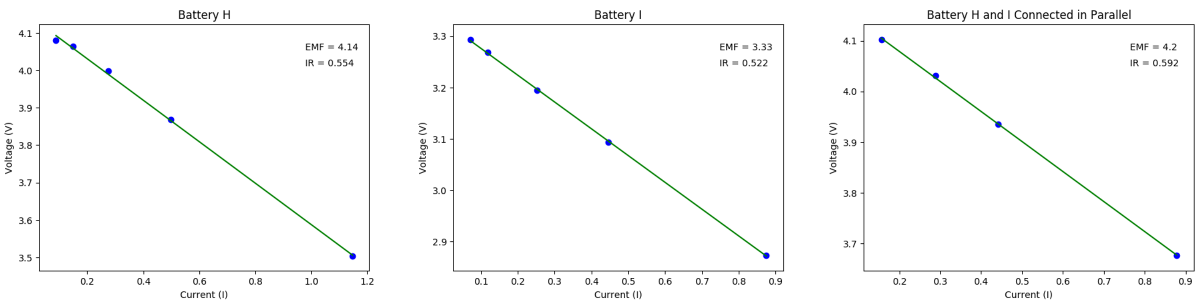
Theoretical EMF	4.12	V	Real EMF	4.27	V	Error in EMF	3.47	%
Theoretical IR	0.247	Ω	Real IR	0.273	Ω	Error in IR	9.67	%

Figure 24: A battery pack consists of battery D and F connected in Parallel



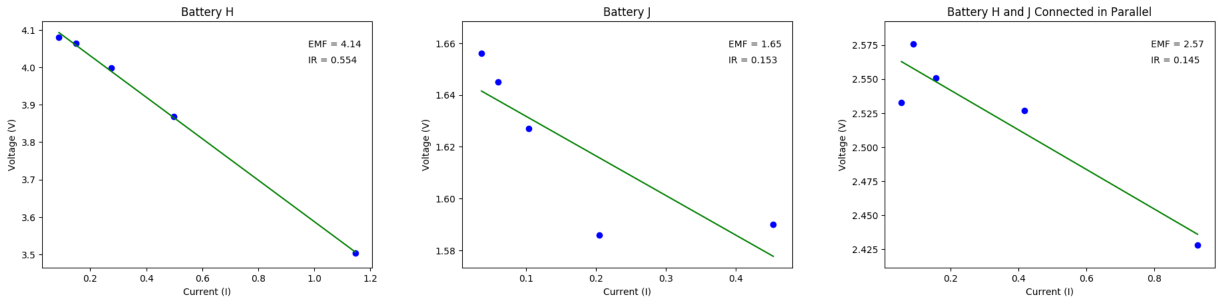
Theoretical EMF	3.16	V	Real EMF	2.87	V	Error in EMF	10.1	%
Theoretical IR	0.227	Ω	Real IR	0.161	Ω	Error in IR	41.1	%

Figure 25: A battery pack consists of battery D and G connected in Parallel



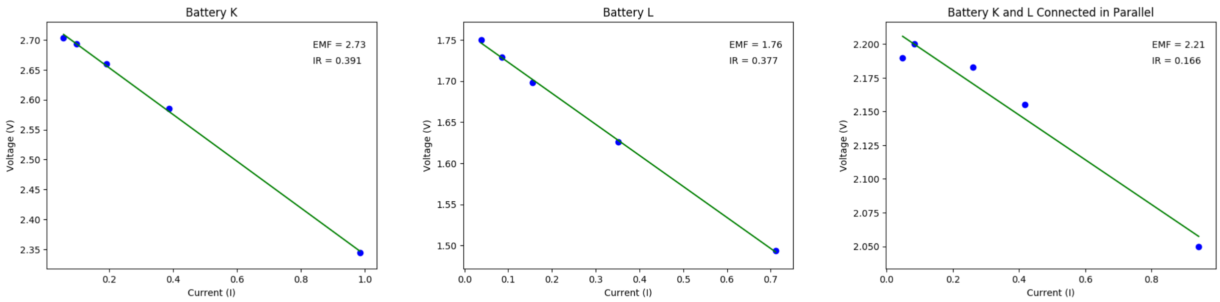
Theoretical EMF	3.72	V	Real EMF	4.20	V	Error in EMF	11.3	%
Theoretical IR	0.269	Ω	Real IR	0.592	Ω	Error in IR	54.6	%

Figure 26: A battery pack consists of battery H and I connected in Parallel



Theoretical EMF	2.19	V	Real EMF	2.57	V	Error in EMF	15.0	%
Theoretical IR	0.120	Ω	Real IR	0.145	Ω	Error in IR	17.7	%

Figure 27: A battery pack consists of battery H and J connected in Parallel

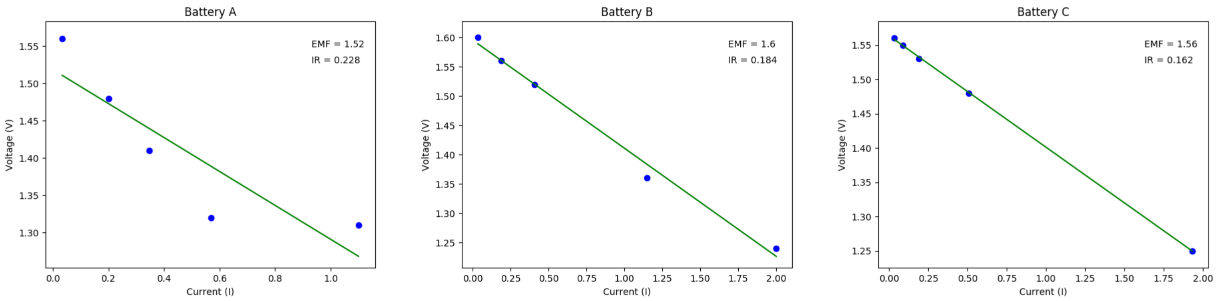


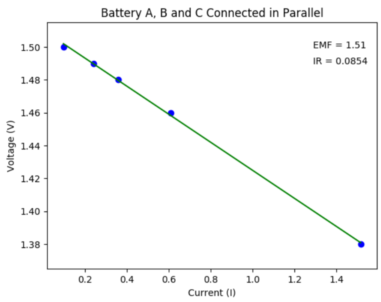
Theoretical EMF	2.24	V	Real EMF	2.21	V	Error in EMF	1.07	%
Theoretical IR	0.192	Ω	Real IR	0.166	Ω	Error in IR	15.7	%

Figure 28: A battery pack consists of battery K and L connected in Parallel

4.2.3 Connection with Multiple Component Batteries

Here the errors in EMF remains to be low with a value of 3 percent. However, notice that the real IR is even less than 0.1 Ω. It's very hard for my apparatus to measure such small resistance accurately. As a result, the errors in IR is still pretty high.





Theoretical EMF	1.56	V	Theoretical IR	0.0626	V
Real EMF	1.51	Ω	Real IR	0.0854	Ω
Error in EMF	3.42	%	Error in IR	26.7	%

Figure 29: A battery pack consists of battery A, B and C connected in Parallel

5 Conclusions and Evaluations

5.1 Conclusions and Implications

Based on my investigations, the equivalent electromotive force and internal resistance of a battery pack depends not only on the arrangement (connection type) of its component cells, but also on the electromotive force and internal resistance of its components cells. Expressions of the equivalent EMF and IR of a battery pack is given below.

When the component batteries of a battery pack are all connected in series, its equivalent EMF and IR could be calculated using equation (14) and (19):

$$\mathcal{E}_{eq} = \sum_{i=1}^{n} \mathcal{E}_i,$$

$$r_{eq} = \sum_{i=1}^{n} r_i.$$

And when its component batteries are all connected in parallel, its equivalent EMF and IR could be calculated using equation (41) and (34):

$$\mathcal{E}_{eq} = r_{eq} \cdot \sum_{i=1}^{n} \frac{\mathcal{E}_i}{r_i},$$

$$r_{eq} = \frac{1}{\sum_{i=1}^{n} \frac{1}{r_i}}.$$

And when its component batteries are connected in combinations of series and parallel connections, its equivalent EMF and IR could be calculated step by step using the above four equations.

In addition to the fulfillment of my curiosity, this investigation provides a fast method to calculate the equivalent EMF and IR of an unknown battery pack according to its component batteries. This method is especially helpful on an unknown battery pack consists of multiple **standard** batteries.

5.2 Error Analysis

5.2.1 Inaccurate Ammeters and Voltmeters

The different errors of the measurement from ammeters and voltmeters lead to different errors of calculated EMF and IR of a battery.

There are one ammeter and one multimeter in my measuring circuit in figure 7, measuring the voltage across the battery and the current in main branch respectively. The readings on the voltmeter is usually more than $1.5V$ with an error of $0.01V$. As a result, the error of voltage is below 0.667 percent.

On the other hand, to protect the circuit, the readings on the ammeter is usually kept below $0.6A$, but it has a larger error of $0.05A$. As a result, the error of current is usually more than 8.33 percent.

Since the error in EMF has same dimensions as the error in voltage, it's generally kept below 0.667 percent. However, since the error in IR has same dimensions as the quotient of voltage and current, its error is more than 9 percent.

5.2.2 Changing EMF and IR of Batteries

The EMF of a battery is dropping after each measurement, since its power is consumed. Similarly, the IR of a battery could be changing after each measurement as well. So it is theoretically impossible to measure the EMF and IR of a battery accurately using the method in my investigation.

Interestingly, the more measurements on a battery I conduct, the more pairs of V and I I get, and the more accurate the EMF and IR of that battery is regressed. But in the mean time, the more power of that battery is consumed, and the more inaccurate the pairs of V and I represent the original EMF and IR of that battery. This might be an uncertainty principle on measuring the EMF and IR of a battery.

5.2.3 Experiment Operation Errors

Consider figure 10, 11, 17, 19, 20, 21, 27, and 29. Unlike the data in other batteries, the pairs of V and I of battery A and J do not fall perfectly linearly on a line. This is likely to be caused by experiment operation errors.

5.2.4 Resistances in Wires

The resistances in wires in the main branch are **not** a source of error. My measurement of the EMF and IR of a battery does not require it to be negligible, because there is an **arbitrary** amount of resistance in the variable resistor. Adding extra resistances in the variable resistor has no effect on the calculated EMF and IR of that battery.

However, the resistances in wires outside the main branch are undesired, and is a source of error, because it influences the accuracy of the readings on voltmeter.

5.2.5 Quality of Linear Regression

The linear regression quality of the five pairs of V and I contributes little to the calculated errors in EMF and IR of a battery. According to figure 30, the r square value of my linear regression hardly influence the errors in predicted EMF and IR of a battery.

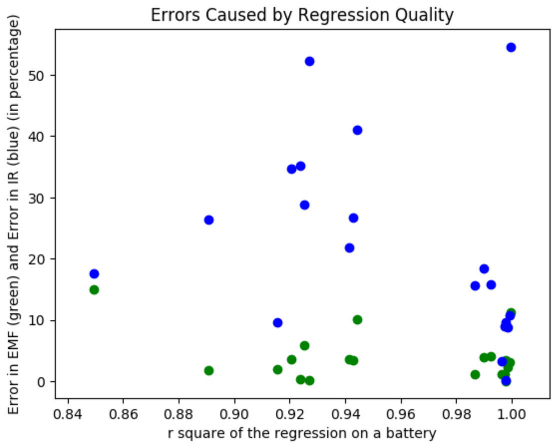


Figure 30: Errors Caused by Regression Quality

5.3 Further Investigations

5.3.1 Complex Connections of Component Batteries

My proposed theories to calculate the equivalent EMF and IR of a battery pack is only limited in the situation, where all the component batteries of that battery pack are connected in combinations of series and parallel connections. However, it still cannot calculate the equivalent EMF and IR of a battery pack if its component batteries are connected in more complex ways. An example is shown in figure 31. A complete theory should provide methods to calculate the equivalent EMF and IR for such batteries.

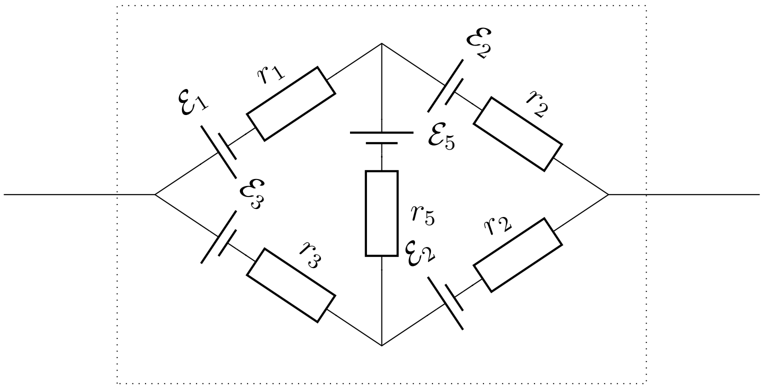


Figure 31: A complex battery pack whose equivalent EMF and IR can't be calculated by my theories

5.3.2 Algorithms to Generate Connection Methods

My investigation only provides a method to calculate the equivalent EMF and IR of a battery pack according to its component batteries. However, reversely, an investigation on algorithms which use standard batteries to assemble battery packs to match a desired EMF and IR would be interesting as well.

References

Oldham, K. T. S. (2008). *The doctrine of description: Gustav kirchhoff, classical physics, and the "purpose of all science" in 19 th-century germany.* University of California, Berkeley.

Tsokos, K. A. (2014). *Physics for the ib diploma coursebook with free online material* (6th ed.). Cambridge University Press.

A Original Data

V_a	I_a	V_b	I_b	V_c	I_c		
1.56	0.034	1.6	0.034	1.56	0.033		
1.48	0.2	1.56	0.188	1.55	0.09		
1.41	0.346	1.52	0.405	1.53	0.19		
1.32	0.569	1.36	1.148	1.48	0.509		
1.31	1.1	1.24	2	1.25	1.933		
V_pab	I_pab	V_pac	I_pac	V_pbc	I_pbc	V_pabc	I_pabc
1.57	0.36	1.44	0.32	1.5	0.098	1.5	0.096
1.56	0.186	1.43	0.344	1.49	0.368	1.49	0.241
1.55	0.288	1.42	0.542	1.48	0.54	1.48	0.359
1.5	0.848	1.4	0.803	1.45	0.986	1.46	0.609
1.39	1.76	1.35	1.49	1.36	2.03	1.38	1.515
V_sab	I_sab	V_sac	I_sac	V_sbc	I_sbc	V_sabc	I_sabc
3.08	0.066	3.06	0.07	3.01	0.064	4.502	0.108
3.06	0.144	3.04	0.15	3	0.129	4.408	0.241
3.03	0.241	3.02	0.202	2.93	0.423	4.318	0.357
2.97	0.448	2.99	0.336	2.81	0.882	4.032	0.949
2.89	0.69	2.83	0.963	2.68	1.15	3.581	2.02
V_d	I_d	V_e	I_e	V_sde	I_sde	V_pde	I_pde
5.46	0.118	4.6	0.108	10.116	0.239	5.222	0.113
5.4	0.262	4.54	0.167	9.925	0.375	5.206	0.18
5.28	0.472	4.47	0.246	9.67	0.573	5.177	0.278
5.1	0.78	4.35	0.389	9.21	0.915	5.06	0.629
4.45	1.85	3.2	2.44	8.16	1.75	4.95	535
		V_f	I_f	V_sdf	I_sdf	V_pdf	I_pdf
		3.066	0.066	8.725	0.187	4.202	0.091
		3.045	0.137	8.45	0.405	4.24	0.146
		2.99	0.266	8.016	0.776	4.186	0.386
		2.9	0.479	6.463	2.188	4.054	0.875
		2.705	0.928	5.769	2.755	3.341	3.38
		V_g	I_g	V_sdg	I_sdg	V_pdg	I_pdg
		1.676	0.036	7.15	0.154	2.811	0.06
		1.653	0.06	7.006	0.28	2.861	0.103
		1.625	0.114	6.731	0.542	2.863	0.175
		1.584	0.211	6.281	0.949	2.864	0.325
		1.35	0.888	5.391	1.822	2.657	1.278
V_h	I_h	V_i	I_i	V_shi	I_shi	V_phi	I_phi
4.08	0.088	3.293	0.072	7.347	0.158	2.713	0.079

4.064	0.148	3.268	0.118	7.237	0.156	4.102	0.156
3.998	0.274	3.194	0.253	7.005	0.471	4.031	0.288
3.869	0.499	3.093	0.446	6.553	0.835	3.935	0.442
3.503	1.148	2.873	0.875	5.837	1.516	3.677	0.878
		V_j	I_j	V_shj	I_sjj	V_phj	I_phj
		1.656	0.036	5.786	0.125	2.533	0.055
		1.645	0.06	5.717	0.191	2.576	0.089
		1.627	0.104	5.583	0.326	2.551	0.156
		1.586	0.205	5.272	0.647	2.527	0.418
		1.59	0.454	4.082	1.897	2.428	0.927
V_k	I_k	V_l	I_l	V_skl	I_skl	V_pkl	I_pkl
2.703	0.058	1.75	0.038	4.441	0.196	2.19	0.048
2.693	0.098	1.729	0.085	4.388	0.162	2.2	0.083
2.66	0.193	1.698	0.155	4.291	0.288	2.183	0.261
2.585	0.388	1.626	0.351	4.061	0.586	2.155	0.418
2.345	0.984	1.494	0.711	3.582	1.217	2.05	0.943

B Python Programs for Data Analysis

B.1 analyze.py

```
import numpy as np
import matplotlib.pyplot as plt

class Battery:

    def __init__(self, name, vList, iList):
        self.name = name
        self.vList = np.array(vList)
        self.iList = np.array(iList)
        self.regress()
        self.draw()
        self.save()

    def regress(self):
        parameters = np.polyfit(self.iList, self.vList, 1)
        self.emf = parameters[1]
        self.ir = - parameters[0]
        corrMatrix = np.corrcoef(self.vList, self.iList)
        r = corrMatrix[0, 1]
        self.r2 = r ** 2

    def draw(self):
        lineEquation = np.poly1d([- self.ir, self.emf])
        linePoints = np.linspace(self.iList[0], self.iList[-1])
        plt.scatter(self.iList, self.vList, color = 'b')
        plt.plot(linePoints, lineEquation(linePoints), color = 'g')
        plt.title('Battery {}'.format(self.name))
        plt.xlabel('Current (I)')
        plt.ylabel('Voltage (V)')
        plt.figtext(0.75, 0.8, 'EMF = %.3g' % self.emf)
        plt.figtext(0.75, 0.75, 'IR = %.3g' % self.ir)

    def save(self):
        nameList = self.name.split()
        if len(nameList) == 1:
            name = nameList[0]
```

```
        else:
            name = nameList[0] + nameList[2] + nameList[-1]
        plt.savefig(name, bbox_inches='tight')
        plt.clf()

class SetofBattery:

    def __init__(self, connectionType, BList, BReal):
        self.connectionType = connectionType
        self.BList = BList
        self.BReal = BReal
        self.calculate()
        self.display()

    def sCalculate(self):
        theoreticalIr = sum([B.ir for B in self.BList])
        theoreticalEmf = sum([B.emf for B in self.BList])
        return (theoreticalIr, theoreticalEmf)

    def pCalculate(self):
        theoreticalIr = 1 / sum([1 / B.ir for B in self.BList])
        theoreticalEmf = theoreticalIr *\
        sum([B.emf / B.ir for B in self.BList])
        return (theoreticalIr, theoreticalEmf)

    def calculate(self):
        if self.connectionType == 's':
            (self.theoreticalIr, self.theoreticalEmf) = self.sCalculate()
        elif self.connectionType == 'p':
            (self.theoreticalIr, self.theoreticalEmf) = self.pCalculate()
        self.irError = abs(self.theoreticalIr - self.BReal.ir)\
        / self.BReal.ir
        self.emfError = abs(self.theoreticalEmf - self.BReal.emf)\
        / self.BReal.emf
        totalR2 = sum([B.r2 for B in self.BList]) + self.BReal.r2
        self.avgR2 = totalR2 / (len(self.BList) + 1)

    def display(self):
        # Error in percentage
        print(self.BReal.name)
        print('%.3g %.3g %.3g %.3g %.3g %.3g'\
```

```
        % (self.theoreticalEmf, self.theoreticalIr,
        self.BReal.emf, self.BReal.ir,
        self.emfError * 100, self.irError * 100))


Va = [1.56, 1.48, 1.41, 1.32, 1.31]
Ia = [0.034, 0.2, 0.346, 0.569, 1.1]
Vb = [1.6, 1.56, 1.52, 1.36, 1.24]
Ib = [0.034, 0.188, 0.405, 1.148, 2]
Vc = [1.56, 1.55, 1.53, 1.48, 1.25]
Ic = [0.033, 0.09, 0.19, 0.509, 1.933]
Vsab = [3.08, 3.06, 3.03, 2.97, 2.89]
Isab = [0.066, 0.144, 0.241, 0.448, 0.69]
Vpab = [1.57, 1.56, 1.55, 1.5, 1.39]
Ipab = [0.36, 0.186, 0.288, 0.848, 1.76]
Vsac = [3.06, 3.04, 3.02, 2.99, 2.83]
Isac = [0.07, 0.15, 0.202, 0.336, 0.963]
Vpac = [1.44, 1.43, 1.42, 1.4, 1.35]
Ipac = [0.32, 0.344, 0.542, 0.803, 1.49]
Vsbc = [3.01, 3, 2.93, 2.81, 2.68]
Isbc = [0.064, 0.129, 0.423, 0.882, 1.15]
Vpbc = [1.5, 1.49, 1.48, 1.45, 1.36]
Ipbc = [0.098, 0.368, 0.54, 0.986, 2.03]
Vsabc = [4.502, 4.408, 4.318, 4.032, 3.581]
Isabc = [0.108, 0.241, 0.357, 0.949, 2.02]
Vpabc = [1.5, 1.49, 1.48, 1.46, 1.38]
Ipabc = [0.096, 0.241, 0.359, 0.609, 1.515]

Vd = [5.46, 5.4, 5.28, 5.1, 4.45]
Id = [0.118, 0.262, 0.472, 0.78, 1.85]
Ve = [4.6, 4.54, 4.47, 4.35, 3.2]
Ie = [0.108, 0.167, 0.246, 0.389, 2.44]
Vf = [3.066, 3.045, 2.99, 2.9, 2.705]
If = [0.066, 0.137, 0.266, 0.479, 0.928]
Vg = [1.676, 1.653, 1.625, 1.584, 1.35]
Ig = [0.036, 0.06, 0.114, 0.211, 0.888]
Vsde = [10.116, 9.925, 9.67, 9.21, 8.16]
Isde = [0.239, 0.375, 0.573, 0.915, 1.75]
Vpde = [5.222, 5.206, 5.177, 5.06, 4.95]
Ipde = [0.113, 0.18, 0.278, 0.629, 0.535]
Vsdf = [8.725, 8.45, 8.016, 6.463, 5.769]
```

```
Isdf = [0.187, 0.405, 0.776, 2.188, 2.755]
Vpdf = [4.202, 4.24, 4.186, 4.054, 3.341]
Ipdf = [0.091, 0.146, 0.386, 0.875, 3.38]
Vsdg = [7.15, 7.006, 6.731, 6.281, 5.391]
Isdg = [0.154, 0.28, 0.542, 0.949, 1.822]
Vpdg = [2.811, 2.861, 2.863, 2.864, 2.657]
Ipdg = [0.06, 0.103, 0.175, 0.325, 1.278]

Vh = [4.08, 4.064, 3.998, 3.869, 3.503]
Ih = [0.088, 0.148, 0.274, 0.499, 1.148]
Vi = [3.293, 3.268, 3.194, 3.093, 2.873]
Ii = [0.072, 0.118, 0.253, 0.446, 0.875]
Vj = [1.656, 1.645, 1.627, 1.586, 1.59]
Ij = [0.036, 0.06, 0.104, 0.205, 0.454]
Vshi = [7.347, 7.237, 7.005, 6.553, 5.837]
Ishi = [0.158, 0.156, 0.471, 0.835, 1.516]
Vphi = [4.102, 4.031, 3.935, 3.677] # First Trial Error
Iphi = [0.156, 0.288, 0.442, 0.878] # First Trial Error
Vshj = [5.786, 5.717, 5.583, 5.272, 4.082]
Ishj = [0.125, 0.191, 0.326, 0.647, 1.897]
Vphj = [2.533, 2.576, 2.551, 2.527, 2.428]
Iphj = [0.055, 0.089, 0.156, 0.418, 0.927]

Vk = [2.703, 2.693, 2.66, 2.585, 2.345]
Ik = [0.058, 0.098, 0.193, 0.388, 0.984]
Vl = [1.75, 1.729, 1.698, 1.626, 1.494]
Il = [0.038, 0.085, 0.155, 0.351, 0.711]
Vskl = [4.441, 4.388, 4.291, 4.061, 3.582]
Iskl = [0.196, 0.162, 0.288, 0.586, 1.217]
Vpkl = [2.19, 2.2, 2.183, 2.155, 2.05]
Ipkl = [0.048, 0.083, 0.261, 0.418, 0.943]

Ba = Battery('A', Va, Ia)
Bb = Battery('B', Vb, Ib)
Bc = Battery('C', Vc, Ic)
Bsab = Battery('A and B Connected in Series', Vsab, Isab)
Bpab = Battery('A and B Connected in Parallel', Vpab, Ipab)
Bsac = Battery('A and C Connected in Series', Vsac, Isac)
Bpac = Battery('A and C Connected in Parallel', Vpac, Ipac)
Bsbc = Battery('B and C Connected in Series', Vsbc, Isbc)
Bpbc = Battery('B and C Connected in Parallel', Vpbc, Ipbc)
```

```
Bsabc = Battery('A, B and C Connected in Series', Vsabc, Isabc) # File name
Bpabc = Battery('A, B and C Connected in Parallel', Vpabc, Ipabc) # File name

Bd = Battery('D', Vd, Id)
Be = Battery('E', Ve, Ie)
Bf = Battery('F', Vf, If)
Bg = Battery('G', Vg, Ig)
Bsde = Battery('D and E Connected in Series', Vsde, Isde)
Bpde = Battery('D and E Connected in Parallel', Vpde, Ipde)
Bsdf = Battery('D and F Connected in Series', Vsdf, Isdf)
Bpdf = Battery('D and F Connected in Parallel', Vpdf, Ipdf)
Bsdg = Battery('D and G Connected in Series', Vsdg, Isdg)
Bpdg = Battery('D and G Connected in Parallel', Vpdg, Ipdg)

Bh = Battery('H', Vh, Ih)
Bi = Battery('I', Vi, Ii)
Bj = Battery('J', Vj, Ij)
Bshi = Battery('H and I Connected in Series', Vshi, Ishi)
Bphi = Battery('H and I Connected in Parallel', Vphi, Iphi)
Bshj = Battery('H and J Connected in Series', Vshj, Ishj)
Bphj = Battery('H and J Connected in Parallel', Vphj, Iphj)

Bk = Battery('K', Vk, Ik)
Bl = Battery('L', Vl, Il)
Bskl = Battery('K and L Connected in Series', Vskl, Iskl)
Bpkl = Battery('K and L Connected in Parallel', Vpkl, Ipkl)

Ssab = SetofBattery('s', [Ba, Bb], Bsab)
Spab = SetofBattery('p', [Ba, Bb], Bpab)
Ssac = SetofBattery('s', [Ba, Bc], Bsac)
Spac = SetofBattery('p', [Ba, Bc], Bpac)
Ssbc = SetofBattery('s', [Bb, Bc], Bsbc)
Spbc = SetofBattery('p', [Bb, Bc], Bpbc)
Ssabc = SetofBattery('s', [Ba, Bb, Bc], Bsabc)
Spabc = SetofBattery('p', [Ba, Bb, Bc], Bpabc)

Ssde = SetofBattery('s', [Bd, Be], Bsde)
Spde = SetofBattery('p', [Bd, Be], Bpde)
Ssdf = SetofBattery('s', [Bd, Bf], Bsdf)
Spdf = SetofBattery('p', [Bd, Bf], Bpdf)
Ssdg = SetofBattery('s', [Bd, Bg], Bsdg)
```

```
Spdg = SetofBattery('p', [Bd, Bg], Bpdg)

Sshi = SetofBattery('s', [Bh, Bi], Bshi)
Sphi = SetofBattery('p', [Bh, Bi], Bphi)
Sshj = SetofBattery('s', [Bh, Bj], Bshj)
Sphj = SetofBattery('p', [Bh, Bj], Bphj)

Sskl = SetofBattery('s', [Bk, Bl], Bskl)
Spkl = SetofBattery('p', [Bk, Bl], Bpkl)

SetList = [Ssab, Spab, Ssac, Spac, Ssbc, Spbc, Ssabc, Spabc,
Ssde, Spde, Ssdf, Spdf, Ssdg, Spdg, Sshi, Sphi, Sshj, Sphj, Sskl, Spkl]

avgR2s = [S.avgR2 for S in SetList]
emfErrors = [S.emfError * 100 for S in SetList]
irErrors = [S.irError * 100 for S in SetList]

plt.scatter(avgR2s, emfErrors, color='g')
plt.scatter(avgR2s, irErrors, color='b')
plt.title('Errors Caused by Regression Quality')
plt.xlabel('r square of the regression on a battery')
plt.ylabel('Error in EMF (green) and Error in IR (blue) (in percentage)')
plt.savefig('regressionQuality', bbox_inches='tight')
plt.clf()
```

B.2 autoInputData.py

```
# Order: IDLE, Atom, Excel

import pyautogui as pag
import time

pag.hotkey('ctrl', 'right')
time.sleep(0.5)
pag.typewrite(' = [')
pag.hotkey('command', 'v')
for i in range(4): pag.press('up')
for i in range(4):
    pag.hotkey('command', 'right')
    pag.typewrite(', ')
```

```
    pag.press('right')
    pag.press('backspace')
pag.hotkey('command', 'right')
pag.press('enter')
pag.hotkey('ctrl', 'right')
```